

6.5 ~~(continued)~~ Average Value

The average y -value of $y = f(x)$ from $x = a$ to $x = b$ is given by

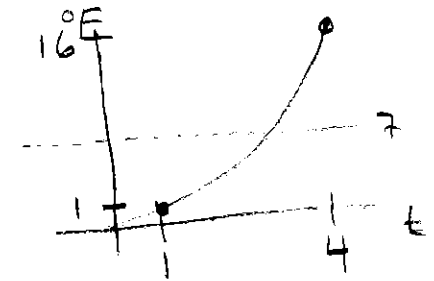
$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

Entry Task:

The formula for the temperature of a particular object is $T(t) = t^2$ degrees Fahrenheit where t is in hours.

Find the average temperature from $t = 1$ to $t = 4$ hours.

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= \frac{1}{4-1} \int_1^4 t^2 dt \\ &= \frac{1}{3} \left[\frac{1}{3} t^3 \Big|_1^4 \right] \\ &= \frac{1}{9} (4^3 - 1^3) \\ &= \frac{1}{9} (64 - 1) \\ &= \boxed{7} \text{ } ^\circ\text{F} \end{aligned}$$



The mean value theorem for integrals:

If $f(x)$ is continuous on from $x = a$ to $x = b$, then there is at least one value $x = c$ at which

$$f(c) = f_{ave}.$$

Example:

Using $T(t) = t^2$ from $t = 1$ to $t = 4$ again.

Find a time at which the temperature is exactly equal to the average value.

$$t^2 \stackrel{?}{=} 7$$

$$t = \sqrt{7} \text{ hours}$$

Average Value Derivation

The average value of the n numbers:

$$y_1, y_2, y_3, \dots, y_n$$

is given by

$$\frac{y_1 + y_2 + y_3 + \dots + y_n}{n} = y_1 \frac{1}{n} + \dots + y_n \frac{1}{n}$$

Goal: We want the average value of all the y -values of some function $y = f(x)$ over an interval $x = a$ to $x = b$.

EX1 Four TEST SCORES

60, 70, 80, 90

$$\text{AVERAGE} = \frac{60 + 70 + 80 + 90}{4} = 75$$

EX) $f(t) = t^2$ $1 \leq t \leq 4$

$n = 6$ subdivisions

$$\Delta t = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2} \Rightarrow 6 = \frac{3}{\Delta t}$$

$$t_0 = 1, t_1 = 1.5, t_2 = 2, t_3 = 2.5, t_4 = 3, t_5 = 3.5, t_6 = 4$$

$$y_1 = f(t_1) = (1.5)^2, y_2 = (2)^2, \dots, y_6 = (4)^2$$

$$\text{AVE} \approx \frac{((1.5)^2 + (2)^2 + \dots + (4)^2)}{6}$$

$$= \frac{1}{6} ((1.5)^2 + (2)^2 + \dots + (4)^2)$$

$$= \frac{\Delta t}{3} ((1.5)^2 + (2)^2 + \dots + (4)^2)$$

$$= \frac{1}{3} ((1.5)^2 \Delta t + 2^2 \Delta t + \dots + 4^2 \Delta t)$$

$$= \frac{1}{3} (f(t_1) \Delta t + f(t_2) \Delta t + \dots + f(t_n) \Delta t)$$

$\leftarrow b-a$

$$\text{AVE VALUE} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$= \frac{1}{b-a} \int_a^b f(x) dx$$

Derivation:

1. Break into n equal subdivisions

$$\Delta x = \frac{b-a}{n}, \text{ which means } \frac{\Delta x}{b-a} = \frac{1}{n}$$

2. Compute y -value at each tick mark

$$y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$$

3. Ave $\approx f(x_1) \frac{\Delta x}{b-a} + \dots + f(x_n) \frac{\Delta x}{b-a}$

$$\text{Average} \approx \frac{1}{b-a} \sum_{i=1}^n f(x_i) \Delta x$$

Thus, we can define

$$\text{Average} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

Which means the exact average y -value of $y = f(x)$ over $x = a$ to $x = b$ is

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

[Ex]

$n=6$

$a \leq x \leq b$

SEE PREVIOUS

PAGE FOR AN
EXAMPLE

7.1 Integration by Parts

Goal: We will reverse the product rule.

Before we start, add these to your basic list of integrals:

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$$

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln |ax + b| + C$$

EX) $\int \sin(5x) dx = -\frac{1}{5} \cos(5x) + C$ CHECK

$\int e^{3x+1} dx = \frac{1}{3} e^{3x+1} + C$ CHECK

$\int \frac{1}{4x+3} dx = \frac{1}{4} \ln |4x+3| + C$ CHECK

Derivation of Integration By Parts

The product rule says:

$$u(x)v'(x) + v(x)u'(x) = \frac{d}{dx}(u(x)v(x))$$

which can be written as

$$\int u(x)v'(x)dx + \int v(x)u'(x)dx = u(x)v(x)$$

Writing this in terms of the differentials:

$$dv = v'(x)dx \text{ and } du = u'(x)dx$$

we have

$$\int u dv + \int v du = uv$$

which we rearrange to get

Integration by Parts formula:

$$\int u dv = uv - \int v du$$

Ex)

$$3x \cdot \cos(x) + 3 \sin(x) = \frac{d}{dx}(3x \sin(x))$$

$$\int 3x \cos(x) dx + \int 3 \sin(x) dx = 3x \sin(x) + c$$

$$\Rightarrow \int 3x \cos(x) dx = 3x \sin(x) - \int 3 \sin(x) dx$$

Example:

$$\int x \cos(8x) dx$$

$$u = x$$

$$dv = \cos(8x) dx$$

$$du = dx$$

$$v = \frac{1}{8} \sin(8x)$$

Step 1: Choose u and dv .

Step 2: Compute du and v .

Step 3: Use formula (and hope)

$$= \frac{1}{8} x \sin(8x) - \int \frac{1}{8} \sin(8x) dx$$

$$= \frac{1}{8} x \sin(8x) - \frac{1}{64} (-\cos(8x)) + C$$

$$= \frac{1}{8} x \sin(8x) + \frac{1}{64} \cos(8x) + C$$

Check!!!

$$\frac{1}{8} \sin(8x) + x \cos(8x) - \frac{1}{8} \sin(8x)$$

SAME ✓✓✓

Example:

$$\int x^2 \ln(x) dx$$

$$u = \ln(x)$$
$$du = \frac{1}{x} dx$$

$$dv = x^2 dx$$
$$v = \frac{1}{3} x^3$$

$$= \frac{1}{3} x^3 \ln(x) - \int \frac{1}{3} x^2 dx$$

$$= \frac{1}{3} x^3 \ln(x) - \frac{1}{9} x^3 + C$$

CHECK!

SAME ✓✓

$$x^2 \ln(x) + \frac{1}{3} x^3 \cdot \frac{1}{x} - \frac{1}{3} x^3$$

Example:

$$\int_1^e x^2 \ln(x) dx$$

$$u = \ln(x) \quad dv = x^2 dx$$
$$du = \frac{1}{x} dx \quad v = \frac{1}{3} x^3$$

$$\frac{1}{3} x^3 \ln(x) \Big|_1^e - \int_1^e \frac{1}{3} x^2 dx$$

$$\left(\left(\frac{1}{3} e^3 \ln(e) \right) - 0 \right) - \frac{1}{9} \left(x^3 \Big|_1^e \right)$$

$$\frac{1}{3} e^3 - \frac{1}{9} (e^3 - 1)$$

$$\frac{1}{3} e^3 - \frac{1}{9} e^3 + \frac{1}{9}$$

$$= \frac{2}{9} e^3 + \frac{1}{9}$$

$$\frac{1}{9} (1 + 2e^3)$$

NOTE: From Previous PAGE

$$\frac{1}{3} x^3 \ln(x) - \frac{1}{9} x^3 \Big|_1^e$$

$$= \left(\frac{1}{3} e^3 \ln(e) - \frac{1}{9} e^3 \right) - \left(0 - \frac{1}{9} \right)$$

$$= \frac{1}{3} e^3 - \frac{1}{9} e^3 + \frac{1}{9}$$

$$= -\frac{2}{9} e^3 + \frac{1}{9}$$

$$= \frac{1}{9} (1 - 2e^3)$$

Notes:

1. The symbols u and v **never** appear in the integration. They are just locations in the formula (no variables are changing, this is not substitution).
2. u and dv completely split up the integrand. **Once you chose u , then dv is everything else.**
3. The goal is to make $\int v du$ “nicer” than $\int u dv$
 - (a) Pick u = “something that gives a derivative that is simpler than the original u ”
 - (b) Pick dv = “something that you can integrate”
 - (c) And hope “ vdu ” is something in our table!

Example:

$$\int x^2 e^{x/2} dx$$

$$u = x^2 \quad dv = e^{\frac{1}{2}x} dx$$
$$du = 2x dx \quad v = 2e^{\frac{1}{2}x}$$

$$= 2x^2 e^{\frac{1}{2}x} - \int 4xe^{\frac{1}{2}x} dx$$

$$u = 4x \quad dv = e^{\frac{1}{2}x} dx$$
$$du = 4 dx \quad v = 2e^{\frac{1}{2}x}$$

$$= 2x^2 e^{\frac{1}{2}x} - (8xe^{\frac{1}{2}x} - \int 8e^{\frac{1}{2}x} dx)$$

$$= 2x^2 e^{\frac{1}{2}x} - 8xe^{\frac{1}{2}x} + 8 \int e^{\frac{1}{2}x} dx$$

$$= 2x^2 e^{\frac{1}{2}x} - 8xe^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C$$

Check!

$$4xe^{\frac{1}{2}x} + x^2 e^{\frac{1}{2}x} - 8e^{\frac{1}{2}x} - 4xe^{\frac{1}{2}x} + 8e^{\frac{1}{2}x}$$

SAME ✓✓✓

Example:

$$\int e^x \cos(x) dx$$

$$\begin{aligned} u &= e^x & dv &= \cos(x) dx \\ du &= e^x dx & v &= \sin(x) \end{aligned}$$

$$= e^x \sin(x) - \int e^x \sin(x) dx$$

$$\begin{aligned} u &= e^x & dv &= \sin(x) dx \\ du &= e^x dx & v &= -\cos(x) \end{aligned}$$

same ✓

$$= e^x \sin(x) - (-e^x \cos(x) - \int -e^x \cos(x) dx)$$

$$\Rightarrow \int e^x \cos(x) = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

$$\Rightarrow (\text{up to a constant}) \quad 2 \int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C_0$$

$$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C$$

$$C = \frac{1}{2} C_0$$

$$= \frac{1}{2} e^x (\sin(x) + \cos(x)) + C$$

CHECK!!!

$$\frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \cos(x) - \frac{1}{2} e^x \sin(x)$$

$e^x \cos(x)$

Example:

$$\int \sin^{-1}(x) dx$$

$$u = \sin^{-1}(x) \quad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = x$$

$$= x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} dx$$

Now WHAT?!?
SUBSTITUTION!

$$u = 1 - x^2$$
$$du = -2x dx$$
$$\frac{1}{-2x} du = dx$$

$$= x \sin^{-1}(x) - \int \frac{x}{\sqrt{u}} \frac{1}{-2x} du \quad u^{-1/2}$$

$$= x \sin^{-1}(x) + \frac{1}{2} \cdot 2 u^{1/2} + C$$

$$= x \sin^{-1}(x) + \sqrt{1-x^2} + C$$

ASIDE IF YOU FORGET $\frac{d}{dx}(\sin^{-1}(x))$

HERE IS HOW WE DERIVED IT IN MATH 124.

$$y = \sin^{-1}(x)$$

$$\Rightarrow \sin(y) = x$$

$$\frac{d}{dx} [\sin(y) = x]$$

$$\Rightarrow \cos(y) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\text{Now } \cos^2(y) = 1 - \sin^2(y)$$

$$\Rightarrow \cos(y) = \pm \sqrt{1 - \sin^2(y)}$$

ON DOMAIN
 $-\pi/2 \leq x \leq \pi/2$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sin(y))^2}} \quad \leftarrow x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

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